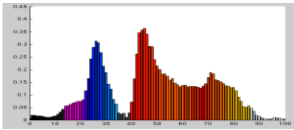


Signals and Systems
Lecture 12: Fourier Series Analysis of Continuous Time Signals- "Frequency Domain Analysis"

Outline
- **Introduction.**
- **Types of Fourier series (Representations).**
- **Eigenfunction property of complex exponential.**
- **Basic Strategy choice.**
- **Fourier analysis.**

Introduction
- **Fourier Series used to analyze periodic signals**
- **The harmonic content of the signals is analyzed with the help of Fourier series (FS).**
- **FS can be developed for CT and DT signals.**

Types of Fourier series (Representations)
1) **Trigonometric FS.**
2) **Compact trigonometric FS, or polar FS.**
3) **Exponential FS (Complex Exponential FS).**

Eigenfunction property of complex exponential
- **Very broad class of signals can be represented using complex exponential.**

$$\Phi_k(t) = e^{j\omega_k t}$$

- **If a linear Time Invariant system is excited by a complex sinusoid, the output response is also a complex sinusoid of the same frequency as the Input; however, the amplitude of such a sinusoid is different from the input amplitude and has a Phase shift.**

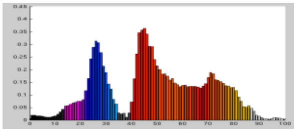
$$x(t) = e^{st}$$

$$x[n] = Z^n$$

s and Z are complex numbers

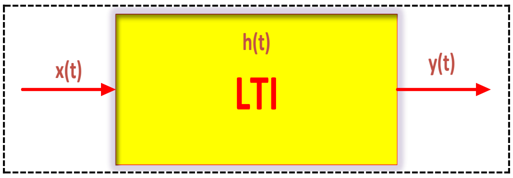
Continuous Time: $e^{st} \rightarrow H(s).e^{st}$

Discrete Time: $Z^n \rightarrow H(z).z^n$



Where the complex amplitude factor $H(s)$ or $H(z)$ will be in general a function of the complex variable **s** or **Z**.

- A **signal** for which the system output is constant times the input is referred to as an Eigenfunction of the system, and the **amplitude factor** is referred as the system's Eigenvalue.
  - ✓ proof :



$$y(t) = \int_{-\infty}^{\infty} h(\tau).x(t-\tau)d\tau$$

For $x(t) = e^{st}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau).e^{s(t-\tau)}d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau).e^{st}.e^{-s\tau}d\tau$$

$$y(t) = e^{st}.\int_{-\infty}^{+\infty} h(\tau).e^{-s\tau}d\tau$$

$H(s) = \int_{-\infty}^{+\infty} h(\tau).e^{-s\tau}d\tau$ ------**Complex constant**

$$y(t) = H(s).e^{st}$$

$H(s) = \int_{-\infty}^{\infty} h(\tau).e^{-s\tau}d\tau$ ------- **Eigenvalue**

e^{st} ---------------------- **Eigenfunction**

- For the analysis of LTI systems, the usefulness of decomposing more **general signals** in terms of eigenfunctions.
  - ✓ **Example:**
- $x(t)$ is a linear combination of three complex exponentials:

$$x(t) = a_1.e^{s_1t} + a_2.e^{s_2t} + a_3.e^{s_3t}$$

From the Eigenfunction property, the response to each separately is:

$$a_1.e^{s_1t} \Rightarrow a_1.H(s_1)e^{s_1t}$$
$$a_2.e^{s_2t} \Rightarrow a_2.H(s_2)e^{s_2t}$$
$$a_3.e^{s_3t} \Rightarrow a_3.H(s_3)e^{s_3t}$$

And from the superposition property:

$$y(t) = a_1.H(s_1)e^{s_1t} + a_2.H(s_2)e^{s_2t} + a_3.H(s_3)e^{s_3t}$$

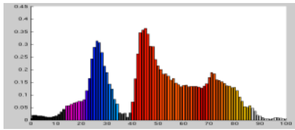
In general:

If

$$x(t) = \sum_k a_k.e^{s_kt}$$

Then the output will be

$$y(t) = \sum_k a_k.H(s_K).e^{s_kt}$$



Basic Strategy choice

If

$$x(t) = a_1.\Phi_1(t) + a_2.\Phi_2(t) + a_3.\Phi_3(t) + \cdots$$

Where

$$\Phi_k(t) \rightarrow \Psi_k(t)$$

And the system is linear

Then

$$y(t) = a_1.\Psi_1(t) + a_2.\Psi_2(t) + a_3.\Psi_3(t) + \cdots$$

, Identical for discrete time Signals:

- ✓ **Examples**

CT system:

$$\begin{Bmatrix} \Phi_k(t) = \delta(t - k\Delta) \\ \Psi_k(t) = h(t - k\Delta) \end{Bmatrix} \quad \text{Convolution Integral}$$

DT system:

$$\begin{Bmatrix} \Phi_k(n) = \delta(n - k) \\ \Psi_k(n) = h(n - k) \end{Bmatrix} \quad \text{Convolution Sum}$$

Other Building Blocks:

$\Phi_k(t) = e^{s_k t}, \rightarrow s_k$ *complex*

$\Phi_k(n) = Z_k^n, \rightarrow Z_k$ *complex*-----**Z- transform**

$s = \sigma + j\omega$ --------**Laplace transform.**

Fourier analysis

Continues Time:

$$S_k = j\omega_k \Rightarrow \Phi_k(t) = e^{j\omega_k t}$$

Discrete Time:

$$|Z_k| = 1 \Rightarrow \Phi_k(n) = e^{j\Omega_k n}$$

- **For linear time Invariant system (Fourier representation)**

$$e^{j\omega_k t} \rightarrow H(\omega_k).e^{j\omega_k t}$$

$H(\omega_k)$ **: Complex factor depends on** ω_k

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau).h(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{j\omega_k(t-\tau)}d\tau$$

$$= e^{j\omega_k t}.\int_{-\infty}^{\infty} h(\tau).e^{-j\omega_k \tau}d\tau$$

$e^{j\omega_k t}$ **E-function**

$H(\omega_k) = \int_{-\infty}^{\infty} h(\tau).e^{-j\omega_k \tau}d\tau$ **E-value**

- **For periodic signals: Fourier series.**
- **For Aperiodic Signals: Fourier Transform.**
- **The frequency response is usually represented in graph by its magnitude and phase as a function of frequency.**